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Analytical Description of Voids in Majumdar-Papapetrou Spacetimes

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Abstract

We discuss new Majumdar-Papapetrou solutions for the 3+1 Einstein-Maxwell equations, with charged dust acting as the external source of the fields. The solutions satisfy non-linear potential equations which are related to well-known wave equations of 1+1 soliton physics. Although the matter distributions are not localised, they present central structures which may be identified with voids.

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1 Introduction

We consider solutions for the Einstein-Maxwell (EM) equations with charged dust acting as the external source of the fields. Our basic equations read

$$G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu}, \tag{1}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu}, \tag{2}$$

where G^{μ}_{ν} and $F^{\mu\nu}$ denote the Einstein and Maxwell tensors, and the total energy-momentum tensor is given by

$$T^{\mu}_{\nu} = E^{\mu}_{\nu} + \rho u^{\mu} u_{\nu}. \tag{3}$$

Here E^{μ}_{ν} is the Maxwell energy-momentum tensor, and the matter term corresponds to dust with energy density ρ and four-velocity u^{μ}. The four-current is defined by the expression

$$J^{\mu} = \sigma u^{\mu}, \tag{4}$$

were σ is the charge density.

We assume that the fluid is static and use the conforstatic metric

$$ds^2 = -V^2\, dt^2 + \frac{1}{V^2}\, h_{ij} dx^i dx^j, \tag{5}$$

where the background metric h_{ij} and V depend only on the space-like coordinates x^1, x^2, x^3. The electrostatic forms of A_{μ} and J^{μ} are given by

$$A_{\mu} = A_0(x^i)\delta^0_{\mu}, \tag{6}$$

$$J^{\mu} = \frac{\sigma(x^i)}{V}\delta^{\mu}_0, \tag{7}$$

with $i = 1, 2, 3$.

Under these conditions, Eq. (2) contains only one non-trivial equation:

$$\frac{1}{\sqrt{h}}\, \partial_j \left(\sqrt{h}\; h^{jk}\, \frac{\partial_k A_0}{V^2} \right) = \frac{4\pi J^0}{V^2}, \tag{8}$$

where h and h^{ij} are the determinant and the inverse of h_{ij}, respectively.

The trace of the Einstein equations is

$$R = -8\pi T, \tag{9}$$

where R denotes the Ricci scalar and $T = T^{\mu}_{\mu}$. We use the decomposition

$$R = V^2 \left[R_h + 2\nabla^2_h \ln V - 2\partial_i \ln V \partial^i \ln V \right]. \tag{10}$$

Here R_h is the Ricci scalar associated to h_{ij}, and ∇^2_h is the three-dimensional Laplacian operator constructed with the same metric. We assume a flat background space, with $R_h = 0$. Therefore, combining Eqs. (9) and (10) we obtain

$$\nabla^2_h \left(\frac{1}{V} \right) = \frac{4\pi T}{V^3}. \tag{11}$$

Following the Majumdar-Papapetrou (MP) procedure,[1, 2] we assume that

$$A_0 = \alpha V, \tag{12}$$

where $\alpha = \pm 1$. As a consequence, the Maxwell equation (8) takes the form

$$\nabla_h^2 \left(\frac{1}{V} \right) = -\frac{4\pi \alpha J^0}{V^2}, \tag{13}$$

which is clearly the same as Eq. (11) whenever the condition

$$T = -\alpha J^0 V \tag{14}$$

holds. This equation can be combined with $J^0 = \frac{\sigma}{V}$ to obtain the alternative expression

$$\sigma = -\alpha T. \tag{15}$$

Since $T = -\rho$ for dust, Eqs. (11) and (15) can be finally expressed as

$$\nabla_h^2 \lambda + 4\pi \rho \lambda^3 = 0, \tag{16}$$

$$\sigma = \alpha \rho, \tag{17}$$

where $\lambda = \frac{1}{V}$. Due to Eqs. (5) and (12), only one Einstein equation is not trivially satisfied.[3, 4] Therefore, solving Eq. (16) is sufficient for finding a solution of the EM equations.

If we identify our flat background space with the Euclidean, three dimensional space and assume $\rho = 0$, then Eq. (16) reduces to the usual Laplace equation $\nabla^2 \lambda = 0$ and the electrovac, multi-black hole solution follows straightforwardly. Assuming spherical symmetry, and using spherical coordinates, we find

$$\lambda = 1 + \frac{m}{r}. \tag{18}$$

In the far-asymptotic region, the behaviour of this solution is approximately given by

$$V \approx 1 - \frac{m}{r}, \quad g_{00} \approx -1 + \tfrac{2m}{r}, \quad A_0 \approx \pm(1 - \frac{m}{r}). \tag{19}$$

The corresponding expression for the electric field is

$$E \approx \frac{q}{r^2}, \tag{20}$$

where

$$q = \pm m. \tag{21}$$

Equation (5) implies that the invariant area of any 2-sphere surrounding the origin is given by $\frac{4\pi r^2}{V(r)^2}$. Therefore, the set $r = 0$, $t = constant$ has a non-zero invariant area given by $4\pi m^2$. In fact, a simple coordinate transform shows that the null hypersurface $r = 0$ is the horizon of the extremal Reissner-Nordström solution. Also, if we define the new radial coordinate $\tilde{r} = -r$ and perform the standard analysis,[5] then we find

that this horizon encloses a point-like, essential singularity placed at $\tilde{r} = m$. In fact, the invariant area vanishes and the scalar $J = F_{\mu\nu} F^{\mu\nu} = \lambda^{-4} \left(\frac{d\lambda}{dr}\right)^2$ blows up at that point.

Equations (16) and (17) were originally discussed by Das [6] in his study of equilibrium configurations of self-gravitating, charged dust. More recently, Gürses [3] has considered non-electrovac solutions when Eq. (16) is linear. This situation corresponds to his choice $\rho = \frac{b^2}{4\pi\lambda^2}$ for constant b. In this case, Eq. (16) admits the particular solution $\lambda = \frac{a \sin br}{r}$ where a is an integration constant. The oscillatory behaviour of this solution implies a geometry with a complicated radial dependence. In fact, the invariant area vanishes for a discrete, infinite set of values of r, and the Ricci scalar $R = \frac{2b^2 r^2}{a^2 \sin^2 br}$ blows up wherever the invariant area vanishes, except for $r = 0$. Other solutions with oscillatory behaviour have been considered by Balakrishna and Wali,[7] Braden and Varela,[8] and Ida.[4] In Section 2 we exploit the general non-linearity of Eq. (16) to obtain new solutions which are free of oscillatory singularities and allow asymptotically flat behaviour.

2 The non-linear models

The non-linear potential equation (16) takes the spherically symmetric form

$$\frac{d^2\lambda}{dr^2} + \frac{2}{r}\frac{d\lambda}{dr} + 4\pi\rho\lambda^3 = 0. \tag{22}$$

Using the new radial coordinate $\tau = \frac{1}{r}$, the same differential equation can be written as

$$\frac{d^2\lambda}{d\tau^2} + \frac{4\pi\rho}{\tau^4}\lambda^3 = 0. \tag{23}$$

If ρ and λ satisfy the condition

$$\rho = \frac{b^2}{4\pi}\frac{\tau^4 \sin\lambda}{\lambda^3}, \tag{24}$$

then (23) finally reduces to the -sine-Gordon equation [9]

$$\frac{d^2\lambda}{d\tau^2} + b^2 \sin\lambda = 0, \tag{25}$$

which has the solutions

$$\lambda^{\pm}(\tau) = 2 \arcsin\left[\tanh\left(\pm b\tau + c\right)\right] + 2n\pi, \tag{26}$$

where n is an arbitrary integer, c is an integration constant, and b is assumed to be positive. We consider only the case $n = 0$. In terms of the original radial coordinate, these solutions read

$$V^{\pm}(r) = \frac{1}{2 \arcsin\left[\tanh\left(\pm\frac{b}{r} + c\right)\right]}. \tag{27}$$

We observe that $V^{\pm}(0)^2$ is finite, so the invariant area vanishes for $r = 0$. Therefore, the set $r = 0$, $t = constant$ is point-like with respect to both solutions. Let us deal with V^+ first. A preliminary numerical study of the invariants J, R, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ suggests that these quantities are bounded for non-negative r, whenever c is positive. If we choose

$$c = \frac{1}{2}\ln\left[\frac{1+\sin(1/2)}{1-\sin(1/2)}\right], \tag{28}$$

then the far-asymptotic behaviour of this solution is given by Eqs. (19), (20), (21) with $m = 2b\cos(1/2)$. Therefore, V^+ is asymptotically flat, exactly as the MP electrovac solution.

The positive definite energy density given by Eq. (24) corresponds to a non-localised matter (and charge) distribution. However, ρ is negligible for $x = \frac{r}{b} \ll 0.2$. For very small x the dimensionless expressions of ρ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ are approximately given by

$$\rho(x) \approx \frac{e^{-c}}{\pi^4}\,\frac{e^{-\frac{1}{x}}}{x^4}, \tag{29}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(x) \approx 3\left(\frac{2}{\pi}\right)^6 e^{-2c}\,\frac{e^{-\frac{2}{x}}}{x^8}. \tag{30}$$

These results imply a very fast decrease of $\rho(x)$ and $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}(x)$ when $x \to 0^+$, and suggest the existence of a void in the innermost region of the asymptotically flat object constructed with V^+. Nevertheless, this interpretation cannot be complete without a better understanding of the point-like set $r = 0$, $t = constant$. A closer look at the singularity contained in this solution is also necessary. We interpret $r = 0$, $t = constant$ as the center of symmetry and observe that the above mentioned invariants are bounded at this point. However, the coordinate transform $\tilde{r} = -r$ reveals the existence of a point-like, essential singularity at $\tilde{r} = \frac{b}{c}$. In fact, the invariant J blows up at this point. The use of V^+ alone may imply the division of the manifold into connected parts, separated by the point-like singularity placed at $r = -\frac{b}{c}$. However, a very different situation comes out when we restrict V^+ to positive values of r and describe the geometry for $r < 0$ with the second solution V^-. Then, a smooth (at least C^1) matching of V^+ and V^- occurs at $r = 0$ and the arising asymptotically flat spacetime seems to be connected and singularity free, with an almost empty region near the center of symmetry. Thus, the joint use of V^+ and V^- provides a simpler description of a MP void. The study of the global structure of these solutions is left as an open problem, which provides motivation for further research work.

Finally, we point out that other exact, non-linear solutions for this theory can be found if we impose different relationships between ρ and λ. For example, the choice

$$\rho = -\frac{b^2}{4\pi}\frac{\tau^4\sin\lambda}{\lambda^3} \tag{31}$$

leads to the sine-Gordon equation

$$\frac{d^2\lambda}{d\tau^2} = b^2\sin\lambda. \tag{32}$$

It has the well-known solutions

$$\lambda^{\pm}(\tau) = 4 \arctan e^{(\pm b\tau + d)}. \tag{33}$$

If we choose $d = \ln[\tan(1/4)]$, then both solutions have asymptotically flat behaviour.

Another example is

$$\rho = \frac{b^2}{4\pi}\left(\lambda - \lambda^3\right). \tag{34}$$

In this case the geometry is determined by the $\lambda\phi^4$ equation

$$\frac{d^2\lambda}{d\tau^2} + b^2\left(\lambda - \lambda^3\right) = 0 \tag{35}$$

which admits the solutions

$$\lambda^{\pm}(\tau) = \tanh\left(\pm\frac{b}{\sqrt{2}}\tau + f\right). \tag{36}$$

The relationship between the 3+1 EM theory and the equations of 1+1 soliton physics deserves a more detailed examination. Possible extensions of this work involve the analysis of dust models for which $\lambda(\tau)$ is a solution of the KdV equation, and the study of the non-linear potential equations arising in higher dimensions.[10]

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[9] B. Saha (gr-qc/9811044) has considered scalar fields satisfying this and other solitonic equations on a static Freedman-Robertson-Walker background.

[10] The electrovac MP solutions in higher dimensions have been discussed by P. Mitra (gr-qc/9908014).